

AN ALGEBRAIC PROOF OF DETERMINANT FORMULAS OF GROTHENDIECK POLYNOMIALS

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ABSTRACT. We give an algebraic proof of the determinant formulas for factorial Grothendieck polynomials obtained by Hudson–Ikeda–Matsumura–Naruse in [4] and by Hudson–Matsumura in [5].

1. DEFINITION AND THEOREM

In [11] and [9], Lascoux and Schützenberger introduced (*double*) *Grothendieck polynomials* indexed by permutations as representatives of K -theory classes of structure sheaves of Schubert varieties in a full flag variety. In [3] and [2], Fomin and Kirillov described them combinatorially in the framework of Yang-Baxter equations. Let $x = (x_1, \dots, x_d)$, $b = (b_1, b_2, \dots)$ be sets of indeterminates. A *Grassmannian permutation* with descent at d corresponds to a *partition* λ of length at most d , *i.e.* a sequence of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_d)$ such that $\lambda_i \geq \lambda_{i+1}$ for each $i = 1, \dots, d-1$. For such permutation, Buch [1] gave a combinatorial expression of the corresponding Grothendieck polynomial $G_\lambda(x)$ as a generating series of *set-valued tableaux*, a generalization of semi-standard Young tableaux by allowing a filling of a box in the Young diagram to be a set of integers. In [14], McNamara gave an expression of *factorial* (*double*) Grothendieck polynomials $G_\lambda(x|b)$ also in terms of set-valued tableaux.

In this paper, we prove the following Jacobi–Trudi type determinant formula for $G_\lambda(x|b)$. For each non-negative integer k and an integer m , let $G_m^{(k)}(x|b)$ be a function of x and b given by

$$G^{(k)}(u) := \sum_{m \in \mathbb{Z}} G_m^{(k)}(x|b) u^m := \frac{1}{1 + \beta u^{-1}} \prod_{i=1}^d \frac{1 + \beta x_i}{1 - x_i u} \prod_{j=1}^k (1 + (u + \beta) b_j),$$

where β is a formal variable of degree -1 and $\frac{1}{1 + \beta u^{-1}}$ is expanded as $\sum_{s \geq 0} (-1)^s \beta^s u^{-s}$. We use the generalized binomial coefficients $\binom{n}{i}$ given by $(1 + x)^n = \sum_{i \geq 0} \binom{n}{i} x^i$ for $n \in \mathbb{Z}$ with the convention that $\binom{n}{i} = 0$ for all integers $i > 0$.

Theorem 1.1. *For each partition λ of length at most d , we have*

$$(1.1) \quad G_\lambda(x|b) = \det \left(\sum_{s \geq 0} \binom{i-d}{s} \beta^s G_{\lambda_i+j-i+s}^{(\lambda_i+d-i)}(x|b) \right)_{1 \leq i, j \leq d}$$

$$(1.2) \quad = \det \left(\sum_{s \geq 0} \binom{i-j}{s} \beta^s G_{\lambda_i+j-i+s}^{(\lambda_i+d-i)}(x|b) \right)_{1 \leq i, j \leq d}.$$

In particular, we have $G_{(k,0,\dots,0)}(x|b) = G_k^{(k+d-1)}(x|b)$.

The formulas (1.1) and (1.2) were originally obtained in the context of degeneracy loci formulas for flag bundles by Hudson–Matsumura in [5] and Hudson–Ikeda–Matsumura–Naruse in [4] respectively. The proof in this paper is purely algebraic, generalizing Macdonald’s argument in [13, (3.6)] for Jacobi–Trudi formula

of Schur polynomials. We use the following “bi-alternant” formula of $G_\lambda(x|b)$ described by Ikeda–Naruse in [6]:

$$(1.3) \quad G_\lambda(x|b) = \frac{\det([x_j|b]^{\lambda_i+d-i}(1+\beta x_j)^{i-1})_{1 \leq i, j \leq d}}{\prod_{1 \leq i < j \leq d} (x_i - x_j)}.$$

Here we denote $x \oplus y := x + y + \beta xy$ and $[y|b]^k := (y \oplus b_1) \cdots (y \oplus b_k)$ for any variable x, y . Note that the Grothendieck polynomial $G_\lambda(x)$ described in [1] coincides with $G_\lambda(x|b)$ by setting $\beta = -1$ and $b_i = 0$.

In the non-factorial case, Jacobi–Trudi type formulas different from the ones in Theorem 1.1 have been also obtained by Lenart in [12] and by Kirillov in [8] (see also the papers [10] by Lascoux–Naruse and [16] by Yeliussizov). It is worth mentioning that in [1] Buch obtained the Littlewood–Richardson rule for the structure constants of Grothendieck polynomials $G_\lambda(x)$, and hence the Schubert structure constants of the K -theory of Grassmannians (see also the paper [7] by Ikeda–Shimazaki for another proof). For the equivariant K -theory of Grassmannians (or equivalently for the factorial Grothendieck polynomials $G_\lambda(x|b)$), the structure constants were recently determined by Pechenik and Yong in [15] by introducing a new combinatorial object called *genomic tableaux*.

2. PROOF OF (1.1)

By (1.3), it suffices to show the identity

$$\frac{\det([x_j|b]^{a_i+d-i}(1+\beta x_j)^{i-1})_{1 \leq i, j \leq d}}{\prod_{1 \leq i < j \leq d} (x_i - x_j)} = \det \left(\sum_{s \geq 0} \binom{i-d}{s} \beta^s G_{a_i+j-i+s}^{(a_i+d-i)}(x|b) \right)_{1 \leq i, j \leq d},$$

for each $(a_1, \dots, a_d) \in \mathbb{Z}^d$ such that $a_i + d - i \geq 0$. For each integer j such that $1 \leq j \leq d$, we let

$$E^{(j)}(u) := \sum_{p=0}^{d-1} e_p^{(j)}(x) u^p := \prod_{\substack{1 \leq i \leq d \\ i \neq j}} (1 + x_i u).$$

We denote $\bar{y} := \frac{-y}{1+\beta y}$. Since $1 + (u + \beta)y = \frac{1 - \bar{y}u}{1 + \beta \bar{y}}$, we have

$$G^{(k)}(u) = \frac{1}{1 + \beta u^{-1}} \prod_{i=1}^d \frac{1 + \beta x_i}{1 - x_i u} \prod_{\ell=1}^k \frac{1 - \bar{b}_\ell u}{1 + \beta \bar{b}_\ell}.$$

Consider the identity

$$(2.1) \quad G^{(k)}(u) E^{(j)}(-u) = \frac{1}{1 + \beta u^{-1}} \frac{1}{1 - x_j u} \prod_{i=1}^d (1 + \beta x_i) \prod_{\ell=1}^k \frac{1 - \bar{b}_\ell u}{1 + \beta \bar{b}_\ell}.$$

$$\sum_{p=0}^{d-1} G_{m-p}^{(k)}(x|b) (-1)^p e_p^{(j)}(x) = x_j^{m-k} \frac{\prod_{\ell=1}^k (x_j - \bar{b}_\ell)}{\prod_{\ell=1}^k (1 + \beta \bar{b}_\ell)} \prod_{\substack{1 \leq i \leq d \\ i \neq j}} (1 + \beta x_i).$$

Since $\frac{y - \bar{b}}{1 + \beta \bar{b}} = y \oplus b$, we have

$$(2.2) \quad \sum_{p=0}^{d-1} G_{m-p}^{(k)}(x|b) (-1)^p e_p^{(j)}(x) = x_j^{m-k} [x_j|b]^k \prod_{\substack{1 \leq i \leq d \\ i \neq j}} (1 + \beta x_i), \quad (m \geq k).$$

Consider the matrices

$$H := \left(\sum_{s \geq 0} \binom{i-d}{s} \beta^s G_{a_i+j-i+s}^{(a_i+d-i)}(x|b) \right)_{1 \leq i, j \leq d} \quad \text{and} \quad M := \left((-1)^{d-i} e_{d-i}^{(j)}(x) \right)_{1 \leq i, j \leq d}.$$

By using (2.2), we find that the (i, j) -entry of HM is

$$(HM)_{ij} = [x_j|b]^{a_i+d-i} (1 + \beta x_j)^{i-d-1} \prod_{1 \leq t \leq d} (1 + \beta x_t).$$

By taking the determinant of HM , the factor $\prod_{1 \leq j \leq d} (1 + \beta x_j)^{-d} \prod_{1 \leq t \leq d} (1 + \beta x_t)^d$ which turns to be 1 comes out, and therefore we obtain

$$\det H \det M = \det \left([x_j|b]^{a_i+d-i} (1 + \beta x_j)^{i-1} \right)_{1 \leq i, j \leq d}.$$

By dividing by $\det M$, we obtain the desired identity since $\det M = \prod_{1 \leq i < j \leq d} (x_i - x_j)$ (see [13, p.42]). \square

3. PROOF OF (1.2)

By (1.3), it suffices to show the identity

$$\frac{\det \left([x_j|b]^{a_i+d-i} (1 + \beta x_j)^{i-1} \right)_{1 \leq i, j \leq d}}{\prod_{1 \leq i < j \leq d} (x_i - x_j)} = \det \left(\sum_{s \geq 0} \binom{i-j}{s} \beta^s G_{a_i+j-i+s}^{(a_i+d-i)}(x|b) \right)_{1 \leq i, j \leq d}.$$

for each $(a_1, \dots, a_d) \in \mathbb{Z}^d$ such that $a_i + d - i \geq 0$. Fix j such that $1 \leq j \leq d$. Let

$$\overline{E}^{(j)}(u) := \sum_{p=0}^{d-1} e_p^{(j)}(-\bar{x}) u^p := \prod_{\substack{1 \leq i \leq d \\ i \neq j}} (1 - \bar{x}_i u).$$

Since $1 + (u + \beta)y = \frac{1 - \bar{y}u}{1 + \beta \bar{y}}$, we have the identity

$$(3.1) \quad G^{(k)}(u) \overline{E}^{(j)}(-u - \beta) = \frac{1}{1 + \beta u^{-1}} \frac{1 + \beta x_j}{1 - x_j u} \prod_{1 \leq \ell \leq k} \frac{1 - \bar{b}_\ell u}{1 + \beta \bar{b}_\ell}.$$

By comparing the coefficient of u^m , $m \geq k$ in (3.1) we obtain

$$(3.2) \quad \sum_{p=0}^{d-1} \sum_{s=0}^p \binom{p}{s} \beta^s G_{m-p+s}^{(k)}(x|b) (-1)^p e_p^{(j)}(-\bar{x}) = x_j^{m-k} \prod_{1 \leq \ell \leq k} \frac{x_j - \bar{b}_\ell}{1 + \beta \bar{b}_\ell} = x_j^{m-k} [x_j|b]^k$$

where the last equality follows from the identity $\frac{x - \bar{y}}{1 + \beta \bar{y}} = x \oplus y$ for any variable x, y . Consider the matrices

$$H' := \left(\sum_{s \geq 0} \binom{i-j}{s} \beta^s G_{a_i+j-i+s}^{(a_i+d-i)}(x|b) \right)_{1 \leq i, j \leq d} \quad \text{and} \quad \overline{M} := \left((-1)^{d-i} e_{d-i}^{(j)}(-\bar{x}) \right)_{1 \leq i, j \leq d}.$$

We write the (i, j) -entry of the product $H' \overline{M}$ as

$$(H' \overline{M})_{ij} = \sum_{p=0}^{d-1} \sum_{s \geq 0} \binom{i-d+p}{s} \beta^s G_{a_i+d-i+s-p}^{(a_i+d-i)}(x|b) (-1)^p e_p^{(j)}(-\bar{x}).$$

By writing $\binom{i-d+p}{s} = \sum_{\ell \geq 0} \binom{i-d}{\ell} \binom{p}{s-\ell}$ using a well-known identity of binomial coefficients and then applying (3.2), we obtain

$$(H' \overline{M})_{ij} = [x_j|b]^{a_i+d-i} (1 + \beta x_j)^{i-1} (1 + \beta x_j)^{1-d}.$$

By taking the determinant of $H' \overline{M}$, we have

$$\det H' \det \overline{M} = \left(\prod_{1 \leq j \leq d} (1 + \beta x_j)^{1-d} \right) \det \left([x_j|b]^{a_i+d-i} (1 + \beta x_j)^{i-1} \right)_{1 \leq i, j \leq d}.$$

Since we have (see [13, p.42])

$$\det \overline{M} = \prod_{1 \leq i < j \leq d} (\bar{x}_j - \bar{x}_i) = \prod_{1 \leq i < j \leq d} \frac{x_i - x_j}{(1 + \beta x_i)(1 + \beta x_j)} = \prod_{1 \leq i \leq d} (1 + \beta x_i)^{1-d} \prod_{1 \leq i < j \leq d} (x_i - x_j),$$

we obtain the desired identity. \square

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